3 - 10 Reduction of order
Reduce to first order and solve, showing each step in detail.

3. $y'' + y' = 0$

Reduction of order is something that Mathematica does not generally need to do.

```
eqn = y''[x] + y'[x] == 0
y′[x] + y″[x] == 0
```

```
sol = DSolve[eqn, y, x]
```

$\{\{y \to \text{Function}[\{x\}, -e^{-x} C[1] + C[2]]\}\}$

```
eqn /. sol // Simplify
{True}
```

5. $y\, y'' = 3\, (y')^2$

```
eqn = y[x] y''[x] == 3 y'[x]^2
y[x] y″[x] == 3 y′[x]^2
```

```
sol = DSolve[eqn, y, x]
```

$\left\{\left\{y \to \text{Function}\left[\{x\}, \frac{C[2]}{\sqrt{2 x + C[1]}}\right]\right\}\right\}$

```
eqn /. sol // Simplify
{True}
```

The text answer is $1 \Big/ \sqrt{c_1 x + c_2}$. So Mathematica and the text answer each have assigned a value to one of their three constants. This leaves leeway for the remaining assignments to be made in such a way that the two solutions become equivalent.

7. $y'' + y'\ ^3 \text{Sin}[y] = 0$

```
ClearAll["Global`*"]
```

This problem is a topsy-turvy little trip with an inverted domain. The substitution $z = y'[x]$ is made. Afterwards there is the form

```
eqn2 = z'[y] z[y] == -z[y]^3 Sin[y]
z[y] z′[y] == -Sin[y] z[y]^3
```

Which can be processed by DSolve into the solution

```
sol2 = DSolve[eqn2, z, y]
```

$$\left\{\{z \to \text{Function}[\{y\}, 0]\}, \left\{z \to \text{Function}\left[\{y\}, \frac{1}{-C[1] - \text{Cos}[y]}\right]\right\}\right\}$$

The above green cell agrees with the text, though the text uses the inverted form of the fractional expression, calling it $\frac{dx}{dy}$. Using the terms of the substitution, the solution checks out.

```
eqn2 /. sol2 // Simplify
{True, True}
```

The next step is to reverse the substitution level by solving again.

```
eqn3 = -x'[y] == C[1] + Cos[y]
-x′[y] == C[1] + Cos[y]
```

```
sol3 = DSolve[eqn3, x, y]
```

```
{{x → Function[{y}, -y C[1] + C[2] - Sin[y]]}}
```

The green cell above matches the final answer in the text, with the provision that the sign on the constant -C[1] is opposite to the constant c_1 in the text. The second use of DSolve also checks out true.

```
eqn3 /. sol3
{True}
```

9. x^2 y'' -5x y' +9 y = 0, $y_1 = x^3$

```
ClearAll["Global`*"]
```

The substitution $y_1 = x^3$ works as advertised as a singular solution. If it is ignored,

```
eqn = x^2 y''[x] - 5 x y'[x] + 9 y[x] == 0
9 y[x] - 5 x y′[x] + x^2 y″[x] == 0
```

then Mathematica comes up with an equivalent solution, so long as C[1] is assigned the value 0 and C[2] is assigned the value $\frac{1}{3}$.

```
sol = DSolve[eqn, y, x]
```

$$\{\{y \to \text{Function}[\{x\}, x^3 C[1] + 3 x^3 C[2] \text{Log}[x]]\}\}$$

The Mathematica solution, neither more nor less general than the text, checks out.

```
eqn /. sol // Simplify
{True}
```

11 - 14 Applications of reducible ODEs

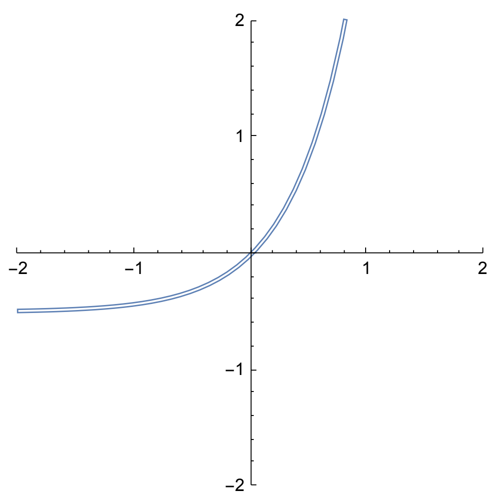
11. Curve. Find the curve through the origin in the xy-plane which satisfies $y'' = 2\, y'$ and whose tangent at the origin has slope 1.

```
In[1]:= ClearAll["Global`*"]

In[2]:= eqn = y''[x] == 2 y'[x]
Out[2]= y''[x] == 2 y'[x]

In[3]:= sol = DSolve[{eqn, y'[0] == 1, y[0] == 0}, y, x]
```

Out[3]= $\{\{y \to \text{Function}[\{x\}, \frac{1}{2}(-1 + e^{2x})]\}\}$

The text solution is $y = c_1\, e^{2x} + c_2$, which is not capable of fulfilling the initial conditions in the problem description unless c_1 and c_2 take on the values $-\frac{1}{2}$ and $\frac{1}{2}$ respectively, i.e. unless the expression equals the green cell above.

In[13]:= Plot[{$\frac{1}{2}(-1 + e^{2x})$, $-0.5 + 0.5\, e^{2x}$}, {x, -2, 2},
AspectRatio → Automatic, PlotRange → {{-2, 2}, {-2, 2}}, PlotStyle →
{{Thickness[0.01]}, {White, Thickness[0.004]}}, ImageSize → 250]

Out[13]=


13. Motion. If, in the motion of a small body on a straight line, the sum of the velocity and acceleration equals a positive constant, how will the distance y[t] depend on the initial velocity and position?

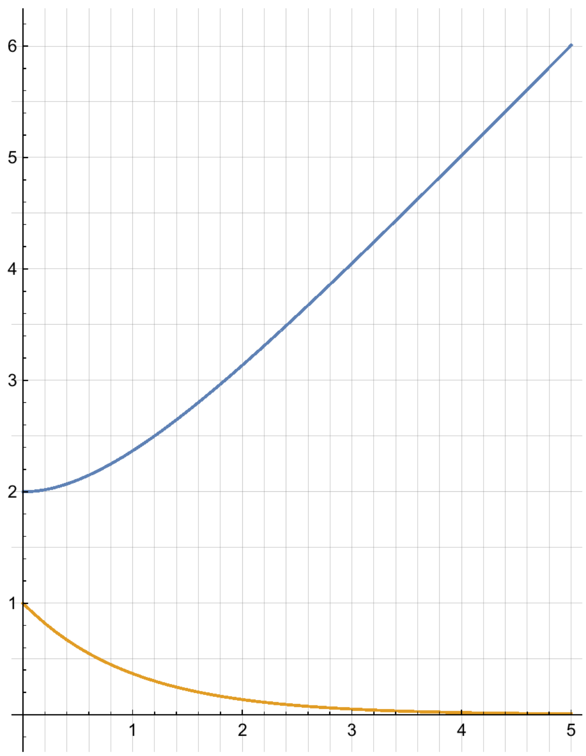
```
ClearAll["Global`*"]
```

First, there is an objection against the statement that the sum of velocity and acceleration equals a constant. The two quantities have different units, so they can't be added. The problem must mean to stipulate that the sum of the coefficients of acceleration and velocity add to a constant. To try to understand this a little bit, I will plot the text answer.

```
y[t_] = c1 e^-t + k t + c2
k t + e^-t c1 + c2
```

The grid squares do not appear as squares, but the axes's major ticks seem to be about equal. The problem is supposed to be about travel along a straight line; here the straight line must be the y-axis. With my choice of c_1, c_2, and $k = 1$, the starting point must be y = 2, and sum of acceleration and velocity must be 1, and the starting velocity must be 1.

```
Plot[{e^-t + t + 1, e^-t}, {t, 0, 5},
 AspectRatio → 1.3, ImageSize → 300, GridLines → All]
```




```
tid = N[Table[{t, e^-t + t + 1}, {t, 0, 15}]]
{{0., 2.}, {1., 2.36788}, {2., 3.13534},
 {3., 4.04979}, {4., 5.01832}, {5., 6.00674}, {6., 7.00248},
 {7., 8.00091}, {8., 9.00034}, {9., 10.0001}, {10., 11.},
 {11., 12.}, {12., 13.}, {13., 14.}, {14., 15.}, {15., 16.}}
```

What can be seen from the two cells below is that by the time t=14, acceleration has nearly disappeared, which means that added velocity is also nearly gone, and the travel velocity is at the rate of the starting velocity.

```
tir = Table[tid[[n]][[2]] - tid[[n]][[1]], {n, 15}]
```

{2., 1.36788, 1.13534, 1.04979, 1.01832, 1.00674, 1.00248,
 1.00091, 1.00034, 1.00012, 1.00005, 1.00002, 1.00001, 1., 1.}

$N[e^{-14}]$

8.31529×10^{-7}

```
y'[t]
```

$k - e^{-t} c_1$

As for the actual problem question, the initial position and initial velocity determine the position of the moving particle in the manner described in the equation for y, governed by the choice of constants.

15 - 19 General solution. Initial value problem (IVP)
(More in the next set.) (a) Verify that the given functions are linearly independent and form a basis of solutions of the given ODE. (b) Solve the IVP. Graph or sketch the solution.

15. 4 *y*" + 25 y = 0, y[0] = 3.0, *y*'[0] = -2.5, Cos[2.5 x], Sin[2.5 x]

```
ClearAll["Global`*"]
```

By inspection, the two trig expressions are independent. (For example, at the time each periodically equals zero, there is no constant which can be multiplied by the zero value to equal the non-zero value of the other.) To test whether they are solutions,

```
eqn = 4 y''[x] + 25 y[x] == 0
```

$25\, y[x] + 4\, y''[x] == 0$

```
sol = DSolve[{eqn, y[0] == 3.0, y'[0] == -2.5}, y, x]
```

$\{\{y \to \text{Function}[\{x\}, 3.\, \text{Cos}[\frac{5x}{2}] - 1.\, \text{Sin}[\frac{5x}{2}]]\}\}$

The solution checks.

```
eqn /. sol // Simplify
```

{True}

The two proposed solutions check.

```
eqn /. Cos[2.5 x] // Simplify
```

ReplaceAll::reps:
 {Cos[2.5x]} is neither a list of replacement rules nor a valid dispatch table, and so cannot be used for replacing. >>

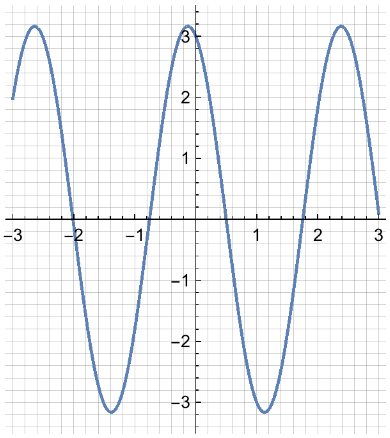
```
True /. Cos[2.5 x]
```

```
eqn /. Sin[2.5 x] // Simplify
```

ReplaceAll::reps: {Sin[2.5x]} is neither a list of replacement rules nor a valid dispatch table, and so cannot be used for replacing. >>

```
True /. Sin[2.5 x]
```

$$\text{Plot}\left[3.\grave{}\ \text{Cos}\left[\frac{5\,x}{2}\right] - 1.\grave{}\ \text{Sin}\left[\frac{5\,x}{2}\right],\ \{x, -3, 3\},\ \text{AspectRatio} \to \text{Automatic},\ \text{ImageSize} \to 200,\ \text{GridLines} \to \text{All}\right]$$



17. $4\,x^2\,y''$ -3 y =0, y(1) = -3, $y'(1) = 0$, $x^{3/2}$, $x^{-1/2}$

```
ClearAll["Global`*"]
```

$$\text{eqn} = 4\,x^2\,y''[x] - 3\,y[x] == 0$$

$$-3\,y[x] + 4\,x^2\,y''[x] == 0$$

```
sol = DSolve[{eqn, y[1] == -3, y'[1] == 0}, y, x]
```

$$\left\{\left\{y \to \text{Function}\left[\{x\}, -\frac{3\,(3 + x^2)}{4\,\sqrt{x}}\right]\right\}\right\}$$

```
eqn /. sol
{True}
```

Although they look a little different due to their format, the green cell above and the text answer are equivalent.

$$\text{PossibleZeroQ}\left[-\frac{3\,(3 + x^2)}{4\,\sqrt{x}} - \left(-0.75\,x^{3/2} - 2.25\,x^{-1/2}\right)\right]$$

```
True
```

Checking the proposed solutions is a little more complicated than usual.

```
d2 = D[x^(3/2), {x, 2}]
```

$$\frac{3}{4\sqrt{x}}$$

```
eqn /. {y[x] -> x^(3/2), y''[x] → d2}
```

True

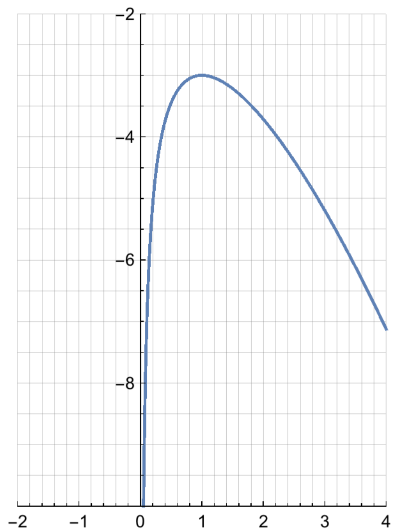
```
d22 = D[x^(-1/2), {x, 2}]
```

$$\frac{3}{4\,x^{5/2}}$$

```
eqn /. {y[x] -> x^(-1/2), y''[x] → d22}
```

True

```
Plot[-(3 (3 + x^2))/(4 Sqrt[x]), {x, -2, 4}, AspectRatio → Automatic,
 ImageSize → 200, GridLines → All, PlotRange → {{-2, 4}, {-10, -2}}]
```



19. $y'' + 2y' + 2y = 0$, $y(0) = 0$, $y'(0) = 15$, e^{-x} Cos[x], e^{-x} Sin[x]

```
ClearAll["Global`*"]
eqn = y''[x] + 2 y'[x] + 2 y[x] == 0;
```

The two given functions are linearly independent by the same argument used for the trig functions in problem 15. Next, the IVP is solved.

```
In[27]:= sol = DSolve[{eqn, y[0] == 0, y'[0] == 15}, y, x]
```

```
Out[27]= {{y → Function[{x}, 15 e^-x Sin[x]]}}
```

Since it does not match the text answer, the solution above is shown in yellow. The Mathematica solution to the IVP is checked:

```
In[17]:= eqn /. sol // Simplify
Out[17]= {True}
```

The first given function is checked to see if it qualifies as a solution.

```
In[18]:= f1[x_] = ⅇ^-x Cos[x]
Out[18]= ⅇ^-x Cos[x]

In[19]:= d1 = D[f1[x], x]
Out[19]= -ⅇ^-x Cos[x] - ⅇ^-x Sin[x]

In[20]:= d2 = D[f1[x], {x, 2}]
Out[20]= 2 ⅇ^-x Sin[x]
```

The first given function is found to qualify.

```
In[21]:= eqn /. {y[x] → f1[x], y'[x] → d1, y''[x] → d2} // Simplify
Out[21]= True
```

The second given function is checked to see if it qualifies as a solution.

```
In[22]:= f2[x_] = ⅇ^-x Sin[x]
Out[22]= ⅇ^-x Sin[x]

In[23]:= d11 = D[f2[x], x]
Out[23]= ⅇ^-x Cos[x] - ⅇ^-x Sin[x]

In[24]:= d22 = D[f2[x], {x, 2}]
Out[24]= -2 ⅇ^-x Cos[x]
```

The second given function is found to qualify.

```
In[25]:= eqn /. {y[x] → f2[x], y'[x] → d11, y''[x] → d22} // Simplify
Out[25]= True
```

The text answer is checked to see if it is a solution.

```
In[29]:= f3[x_] = 15 ⅇ^-x - Sin[2.5 x]
Out[29]= 15 ⅇ^-x - Sin[2.5 x]

In[30]:= d111 = D[f3[x], x]
Out[30]= -15 ⅇ^-x - 2.5 Cos[2.5 x]

In[31]:= d222 = D[f3[x], {x, 2}]
Out[31]= 15 ⅇ^-x + 6.25 Sin[2.5 x]
```

The text answer is found to not qualify as a solution.

In[32]:=
```
eqn /. {y[x] → f3[x], y'[x] → d111, y''[x] → d222} // Simplify
```

Out[32]=
```
1. Cos[2.5 x] == 3. e^-x + 0.85 Sin[2.5 x]
```

There appears to be a typo in the text answer. A plot of the Mathematica solution is made.

In[26]:=
```
Plot[15 e^-x Sin[x], {x, -1, 2}, AspectRatio → Automatic,
 ImageSize → 150, GridLines → All, PlotRange → {{-1, 2}, {-1, 5}}]
```

Out[26]=